Can (noncommutative) geometry accommodate leptoquarks?

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Abstract

We investigate the geometric interpretation of the Standard Model based on noncommutative geometry. Neglecting the S_0-reality symmetry one may introduce leptoquarks into the model. We give a detailed discussion of the consequences (both for the Connes-Lott and the spectral action) and compare the results with physical bounds. Our result is that in either case one contradicts the experimental results.

1 Introduction

In the past years the Standard Model has been an object of investigations directed towards its geometrical foundation within the framework of non-commutative geometry (See [1, 2] and the references therein). The main idea behind this concept is to generalize the notion of manifold and differential structures to an algebraic setup and it appears that one may interpret the particle content of the Standard Model as related to a *discrete noncommutative manifold*. Using this input it is possible to derive the complete classical action, the weak hypercharges and all couplings between fermions

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and bosons. Moreover, the Higgs is naturally explained as a gauge boson related to the discrete differential structure, the symmetry breaking Higgs potential appearing as the Yang-Mills action for this gauge field.

Some further speculations concern mass relations [3] or the quantum group symmetry structure behind this model [1, 4]. These could yield promising results, which might be easy to verify experimentally.

The reported anomaly in high-Q $e^{\pm}p$ collisions at HERA has aroused interest as a possible signal of physics beyond the Standard Model [5, 6]. While these results have yet to be confirmed by other experiments some explanations have already been proposed. Generally, it seems that within the models which are based on a $SU(3) \times SU(2) \times U(1)$ gauge invariant Lagrangian only scalar leptoquarks of certain type can explain the data [6].

In this letter we would like to discuss the predictions and constraints on scalar leptoquarks which one gets from the noncommutative geometry description of the Standard Model. Let us note that contrary to some earlier results, terms which can break the $SU(3)$ symmetry are admissible in the model, provided that one does not impose the so-called S_0 symmetry. We shall present the general construction scheme and suggest what might prevent the breaking of the color symmetry even though leptoquarks are present.

Let us stress that the calculation of the differential calculus for a model of such complexity has not been considered before and is an interesting topic in itself. Details are conveyed to the Appendix, together with other technical observations concerning the model.

2 The Standard Model in Noncommutative Geometry

The crucial role in the model is played by the algebra $\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ and its graded representation space $\mathcal{H}$, a Hilbert space containing all particles of the Standard Model. The algebra acts on the elements of $\mathcal{H}$ from the left and from the right, as shown in the following table :

	$\mathbb{C}^*$	$\mathbb{C}$	$\mathbb{H}$	$M_3(\mathbb{C})$
$\mathbb{C}^*$		e_R		d_R
$\mathbb{C}$	$\bar{e}_R$		$\bar{e}_L, \bar{\nu}_L$	u_R
$\mathbb{H}$		e_L, ν_L		u_L, d_L
$M_3(\mathbb{C})$	$\bar{d}_R$	$\bar{u}_R$	$\bar{u}_L, \bar{d}_L$	

Here, the convention is chosen such that the components of the algebra act from the left along the rows and conjugated elements act from the right along the columns. $\mathbb{C}^*$ means that the complex numbers act by multiplication with $\bar{z}$ instead of z. The left-handed particles are in doublets, on which quaternions act by their 2-dimensional representation, and each quark has additional color indices on which $M_3(\mathbb{C})$ acts.

The model has two symmetries, γ, which has values $+1$ for right-handed particles and -1 for the left-handed, and an antilinear isometry J, which exchanges particles and antiparticles.

To construct the total Hilbert space of fundamental fermions one has to take the tensor product with the bispinor bundle on the 4-dimensional manifold (for problems associated with the doubling of particles see [7]).

3 Dirac operator and particle interactions.

As in the case of gravity, the interactions between particles occur due to the presence of the generalized Dirac operator. The additional principle of gauge invariance requires the existence of bosonic gauge fields, whose dynamics is set by the Yang-Mills action.

The coupling of the gauge boson fields is defined directly by the structure of the Dirac operator. For the whole theory, it consists of two parts, the usual Dirac operator $\gamma^\mu \partial_\mu$ on the 4-dimensional manifold and the discrete Dirac operator D_F which is a linear operator on $\mathcal{H}$, satisfying certain symmetry restrictions.

The gauge fields associated with the first one are the *usual* vector bosons $(W^\pm, Z, \gamma, G^a)$. The new phenomenon is the appearance of bosons associated with the discrete part.

Discrete noncommutative manifolds [8, 9] allow only for Dirac operators which link objects in the same row (in the same column) and connect particles of different chirality.

From the above table, we immediately get the possible actions of D_F, between right-handed leptons (quarks) and the left-handed leptons (quarks), and of course, similarly, for the antiparticles:

$$D_F: \quad e_R \longleftrightarrow (e_L, \nu_L)$$
$$D_F: \quad u_R, d_R \longleftrightarrow (u_L, d_L)$$

These are the only possibilities for the Dirac operator acting between particles only (the conjugate would be among antiparticles). The principle of the S_0-reality condition [1, 9] was, shortly speaking, to enforce that this is

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the case. Then there is no direct coupling between leptons and quarks in the model and the $SU(3)$ symmetry remains unbroken. The gauge potential induced by this part is naturally interpreted as the Higgs field and one can easily see that it couples to leptons and quarks as expected.

However, it is easy to notice that if we do not require S_0-reality, the following chain of links is allowed:

$$D: \quad e_R \longleftrightarrow (e_L, \nu_L) \longleftrightarrow \bar{u}_R.$$

The existence of such a part of the Dirac operator has profound consequences for the physical content of the model. First, it allows for the existence of gauge bosons which couple directly to left-handed leptons and the right-handed *up* antiquark. Second, the resulting action possibly contains terms which break exact $SU(3)_c$, as feature which, of course, is unwanted. The new bosons would have the properties of scalar leptoquarks.

There is neither an experimental nor a theoretical reason to exclude such particles from the model. Also, we have not found a compelling mathematical (topological) advantage of this requirement.

Before we present the action and discuss whether one can consistently include such leptoquarks, let us point out what type of leptoquark is admissible. As shown above, the model leaves room only for a scalar particle which couples to right-handed u antiquarks and the left-handed lepton doublet (of course, there exists also the charge conjugated coupling).

Therefore, within this model one can make a strong prediction concerning the existence of *allowed* couplings, which could be tested experimentally.

4 Construction of the action

In this section we shall outline the calculation of the action which one obtains for the leptoquarks. We shall not be interested in the couplings of fermions or other gauge bosons to leptoquarks, as they will not differ from the usual gauge-invariant terms. Also the mass- or coupling constant relations, which might possibly appear we leave for further study. Our primary interest is to verify whether using the general principle of Yang-Mills theory one may obtain a consistent model without $SU(3)$ symmetry breaking. Regarding more technical details we refer the reader to the appendix.

Models based on noncommutative geometry usually are constructed using the Connes-Lott action principle [10] for gauge theories, which is the noncommutative extension of the Yang-Mills action. More recently a *spectral*

action principle was proposed [11]. While for classical differential manifolds they yield the same result, they differ significantly when one includes the noncommutative discrete structure. Here we shall briefly sketch both approaches.

4.1 Connes-Lott Action

The main steps in constructing the Connes-Lott action is the determination of the differential structure and the scalar product, in particular for the bimodule of two-forms. This is usually done as follows, first an algebra $\Omega_D(\mathcal{A})$ is constructed, as the subalgebra of operators generated by $\mathcal{A}$ and commutators $[D, a]$ for $a \in \mathcal{A}$. In a natural way it is an image of the *universal differential algebra*, $\Omega_u(\mathcal{A})$, which consists of elements $a_0 da_1 \ldots da_k$, $a_i \in \mathcal{A}$, with the differential map d, $d(a_0 da_1 \ldots da_k) = da_0 da_1 \ldots da_k$, satisfying the usual Leibniz rule and $d^2 = 0$. Now, using the map $\pi : \Omega_u(\mathcal{A}) \rightarrow \Omega_D(\mathcal{A})$ one can find a differential algebra $\Omega(\mathcal{A})$ such that the kernel of the differential map $\pi_d : \Omega_u(\mathcal{A}) \rightarrow \Omega(\mathcal{A})$ contains the kernel of the map π. The construction is unique if one postulates that the obtained differential algebra $\Omega(\mathcal{A})$ is maximal. Since the algebra $\Omega_D(\mathcal{A})$ is equipped with a natural scalar product (as a subalgebra of the operator algebra), one has only to choose an appropriate embedding of $\Omega(\mathcal{A})$ in $\Omega_D(\mathcal{A})$, this is usually done as an orthogonal embedding, i.e., the image of a form must be orthogonal to the kernel of the projection on differential forms. For details see [2].

As we are interested here in qualitative answers, we restrict ourselves only to some part of the Hilbert space and Dirac operator, which describes the leptons and the right-handed antiquark $\bar{u}$. Here both the Higgs and leptoquark sector play a role. What we leave out is the quark sector where only the standard Higgs appears.

The algebra $\mathcal{A} = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ acts on the chosen sector of the Hilbert space $\mathcal{H} = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{C}^3$, as $\bar{z} \oplus q \oplus m$, $z \in \mathbb{C}, q \in \mathbb{H}, m \in M_3(\mathbb{C})$.

The allowed Dirac operator is represented as :

$$
\begin{pmatrix}
0 & a & 0 \\
a^\dagger & 0 & b \\
0 & b^\dagger & 0
\end{pmatrix},
$$

where $a : \mathbb{C}^2 \rightarrow \mathbb{C}$ and $b : \mathbb{C}^3 \rightarrow \mathbb{C}^2$ are a priori arbitrary (complex) linear operators. We can use our knowledge of the Standard Model to associate the masses of the leptons with a.

The gauge bosons related with the discrete differential structure appear to be represented by a doublet Φ of complex fields (the Higgs) and six (a

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doublet whose components are triplets with regard to $SU(3)$) complex fields, which we shall call Ψ, having the following gauge transformation rules:

$$\Phi' = U_1 \Phi U_2^\dagger, \quad \Psi' = U_2 \Psi U_3^\dagger,$$

where U_1, U_2, U_3 denote, respectively, $U(1), SU(2), SU(3)$ transformations.

Details of the action calculation are given in the Appendix. In general, there will be three terms which contribute under certain circumstances to the action:

1. Higgs self-interaction term:

$$a \left(\Phi \Phi^\dagger - 1 \right)^2 a^\dagger,$$

2. Higgs and leptoquark self-interaction term:

$$\sim \mathrm{Tr} \left(\Phi^\dagger a^\dagger a \Phi - a^\dagger a + \Psi \Psi^\dagger - bb^\dagger \right)^2,$$

3. Higgs - leptoquarks coupling:

$$a \left(\Phi \Psi \Psi^\dagger \Phi^\dagger \right) a^\dagger.$$

The first one is the well-known symmetry breaking Higgs potential, the second one generates both $SU(3)$ symmetry breaking as well as mass terms for Higgs and leptoquarks, whereas the third one (which occurs only under the condition $ab = 0$) gives a contribution to the mass-terms for leptoquarks.

Of course, physically, the most interesting situation would be to have no color symmetry breaking but massive leptoquarks. We shall briefly discuss all possibilities within the model and slight extensions thereof:

- If the potential term exists there is $SU(3)$ spontaneous symmetry breaking. In this situation the gluons become massive. Moreover, there will be direct interaction terms between leptons and quarks (arising from the vacuum expectation value of Ψ), which lead to lepton and baryon number violation. Note that since Ψ carries lepton and baryon numbers the vacuum would also have such quantum numbers. Such a model is phenomenologically unacceptable.

- If the leptoquarks couple *diagonally* to all families and, moreover, the couplings are identical, then the leptoquark part of the curvature form,

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i.e. the second term, vanishes. However, in such a situation either both leptoquarks or at least one component must be massless. This seems to be excluded experimentally because massless leptoquarks should be observable in atomic spectroscopy.

- If the differential structure on the algebra *does not* coincide with the one introduced[1] it is well possible that even without the condition on diagonality, we would have no potential term. The problem with the masses remains.

- We should mention that the extension of the model including massive neutrinos [2], would not solve the mass problem. If one still assumes a diagonal coupling of the leptoquark to the families, it would decouple from the Higgs and thus remain massless. (The condition $ab = 0$ will not be fulfilled in this situation with $b \neq 0$.)

4.2 The spectral action

We devote a separate paragraph to the action obtained using the spectral principle of Connes [11, 1]. Here, the action is related to the eigenvalues of D^2, D being the Dirac operator. Such an action, applied to the tensor product of classical geometry with the considered noncommutative manifold yields both Yang-Mills and gravity terms. Although the principle includes a large number of free parameters and although one additionally obtains some unrealistic higher-order terms for gravity, the Yang-Mills-Higgs action for the Standard Model and fermionic actions are recovered correctly. However, the whole picture is changed dramatically because the discrete differential structure becomes irrelevant and,[3] for instance, one may not speak any longer of the Higgs potential term as arising from the curvature of the Higgs connection.

 Nevertheless it is worth investigating what the action including the leptoquarks would look like. Shortly speaking one obtains similar terms as in the Connes-Lott model, the difference being, however, in the much smaller number of free parameters. The most important difference comes , of course, from the fact that one does not use Ω^2: one can not get rid of the potential terms for the leptoquark by assuming a diagonal coupling to the families.

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[1]One may, for instance, argue that it comes from the quantum group structure related with the algebra [12].

[2]Note that here Poincaré duality would not be satisfied!

[3]In fact, only Ω^1 plays a role.

Generally one gets Higgs and leptoquark self-interaction terms as well as a Higgs-leptoquark interaction term.

The resulting potential can be written as

$$V(\vec{\psi}, \phi) = \alpha \left(|\vec{\psi_1}|^4 + |\vec{\psi_2}|^4 + 2|\vec{\psi_1} \cdot \vec{\psi_2}|^2 \right) - \frac{\mu^2}{2} \left(|\vec{\psi_1}|^2 + |\vec{\psi_2}|^2 \right)$$
$$+ \delta|\vec{\psi_1}|^2 \phi^2 + \lambda\phi^4 - \frac{\mu^2}{2}\phi^2.$$

where all parameters are positive and $\alpha > \lambda$.

Note that $\vec{\psi_2}$ decouples from the Higgs ϕ. Moreover, μ^2 being positive, there is no mass term for this field if we assume that the vacuum expectation value of $\vec{\psi_1}$ vanishes. Even more so, the minimum of this potential clearly requires $\vec{\psi_2}$ and $\vec{\psi_1}$ to be orthogonal. But then $\vec{\psi_2}$ decouples from all other fields. The minimum for its potential is provided if

$$|\vec{\psi_2}|^2 = \frac{\mu^2}{4\alpha},$$

it breaks $SU(3)$ spontaneously and violates lepton and baryon number conservation.

The vacuum expectation value of $\vec{\psi_1}$ is zero, while $\phi_0^2 = \frac{\mu^2}{4\lambda}$ as usual. Additionally, although less catastrophic, it turns out that one can not adjust the couplings of the leptoquark to the fermions to be consistent with the present experimental bounds.

5   Conclusions

The main conclusion of this Letter is that it is possible to accommodate leptoquarks in the usually assumed model based on noncommutative geometry. The price one pays is the breaking of the so-called S_0-reality condition.

Taking such a model as input one obtains a stringent prediction concerning the possible type of leptoquarks: if they exist only couplings between left-handed leptons and a right-handed $\bar{u}$ antiquark can exist. As a consequence there cannot be any anomaly in $e^- p \to eX$ events at HERA. Moreover, the missing evidence of a charged current signal $e^+ p \to \nu X$ favors scalar leptoquarks. As far as other bounds are concerned this type of leptoquark seems not to be excluded experimentally.

However, the construction of the action leads to several problems and it is difficult to construct a consistent model without the breaking of color symmetry or getting massless leptoquark states. The problem with breaking

color symmetry in the model can be dealt with if one uses the Connes-Lott action. Bounds on flavor mixing which suggest that leptoquarks couple almost diagonally to families, do not contradict the model. On the contrary, diagonal or almost diagonal coupling seems to be significant for the strong symmetry to remain unbroken.

The main problem are the leptoquark masses, atomic spectroscopy seems to exclude massless leptoquarks, moreover, a bound arising from the ρ-parameter suggests that the two isospin components be nearly mass degenerate. Both, the Connes-Lott action as well as the spectral action contradict this experimental data. Of course, one cannot exclude that the whole principle of constructing the action must be modified in which case the results might change dramatically.

On the other hand, it might be possible to find a theoretical reason that excludes the appearance of leptoquarks. One could, for instance, *require* the mentioned S_0-reality. Another idea, which we find more attractive, is to enforce the absence of the coresponding part of the Dirac-operator by a *principle*, which is directly related to the structure of differential calculus and symmetries [8, 12].

Appendix

We discuss here the technical details of the model construction. Before we turn to the noncommutative differential calculus it is instructive to examine the gauge invariant terms, which can possibly appear in the potential for the Higgs and the leptoquarks. Clearly, only the couplings of the two types of scalar fields are interesting. Let us denote the six components of the scalar field Ψ as ψ_k^i $i = 1, \ldots, 3$ $k = 1, 2$ and the two components of the Higgs Φ as ϕ_k $k = 1, 2$.

Note that, due to the freedom of a $SU(2)$-gauge transformation, we can assume

$$\phi_2 = 0, \qquad \phi = \phi_1 \ \text{real}.$$

There are two different possibilities:

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$$\Phi^\dagger \Phi \operatorname{Tr}\left(\Psi^\dagger \Psi\right) = \phi^2 \left(\left|\vec{\psi_1}\right|^2 + \left|\vec{\psi_2}\right|^2\right)$$

In this case the vacuum expectation value of the Higgs will lead to a mass term for all six components of the leptoquark. The masses are degenerate. Unfortunately, this term will not appear in the models, which are based on noncommutative geometry.

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$$\Phi^\dagger \Psi \Psi^\dagger \Phi = \operatorname{Tr}\left(\Psi^\dagger \Phi \Phi^\dagger \Psi\right) = \operatorname{Tr}\left(\Psi \Psi^\dagger \Phi \Phi^\dagger\right) = \phi^2 \left|\vec{\psi_1}\right|^2$$

Here the mass terms for the components $\vec{\psi_2}$ will not get a contribution from the Higgs' expectation value.

A The discrete differential structure

Given the representation and the Dirac operator we can now construct the differential algebra, following the usual procedure [3, 8].

$\Omega^1(\mathcal{A})$ is isomorphic to a bimodule of operators on $\mathcal{H}$ of the following form:

$$\begin{pmatrix} 0 & \bullet & 0 \\ \bullet & 0 & \bullet \\ 0 & \bullet & 0 \end{pmatrix},$$

where the possible nonvanishing entries (bullets) can be arbitrary. Left- and right multiplication by the elements of $\mathcal{A}$ is the usual matrix multiplication.

$\Omega^2(\mathcal{A})$ is a quotient of $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ by the subbimodule generated by the commutators $[r, D^2]$ for $r \in \mathcal{A}$.

It is clear that $\Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ is represented on $\mathcal{H}$ as operators:

$$\begin{pmatrix} \bullet & 0 & \bullet \\ 0 & \bullet & 0 \\ \bullet & 0 & \bullet \end{pmatrix},$$

again with arbitrary entries.

The interesting part is the subbimodule that we have to quotient out. D^2 becomes:

$$\begin{pmatrix} aa^\dagger & 0 & ab \\ 0 & a^\dagger a + bb^\dagger & 0 \\ b^\dagger a^\dagger & 0 & b^\dagger b \end{pmatrix},$$

It is clear that we have to consider two separate situations:

- $ab = 0$ and $a \neq 0$ and $b \neq 0$ [4]

- $ab \neq 0$

We begin with the former.

For $ab = 0$ we have to distinguish two cases, depending on the value of $a^\dagger a + bb^\dagger$. Before we do so, let us make a remark concerning decomposition of $M_2(\mathbb{C})$ as a bimodule of $\mathbb{H}$.

Remark: $M_2(\mathbb{C}) \sim \mathbb{H} \oplus \mathbb{H}$ as a bimodule over $\mathbb{H}$. Unless an element of $M_2(\mathbb{C})$ belongs to one of these components of the direct sum it generates the whole of $M_2(\mathbb{C})$.

- $a^\dagger a + bb^\dagger$ is proportional to 1

  Then, this part of D^2 commutes with everything and therefore the bimodule generated by $[r, D^2]$ is of the form:

  $$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_3(\mathbb{C}) \end{pmatrix}.$$

- $a^\dagger a + bb^\dagger$ is not proportional to 1.

  Then only the traceless part of $a^\dagger a + bb^\dagger$ contributes to the bimodule of the junk, its most general form is:

  $$\begin{pmatrix} r & z \\ \bar{z} & -r \end{pmatrix},$$

  and one can verify that it is of the form qi, where q is a quaternion given by the pair (r, z) and i is the matrix $\mathrm{diag}(1, -1)$, which generates the subbimodule isomorphic to $\mathbb{H}$ in $M_2(\mathbb{C})$. Therefore, the ideal generated by $[a, D^2]$ looks like:

  $$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \mathbb{H} & 0 \\ 0 & 0 & M_3(\mathbb{C}) \end{pmatrix}.$$

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[4]Of course, $a \neq 0$ as we know from the phenomenology of the SM, if $b = 0$ then the whole discussion reduces to the previously widely discussed models [!]

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If $ab \neq 0$, the whole procedure is quite similar, however, additionally we have to take care of the off-diagonal entries of D^2. Remember [8] that apart from the bimodule generated by $[a, D^2]$, we would have contributions from elements of the form $\sum_i a_i(\Xi - \xi\xi)b_i$ for any a, b such that $a_i \xi b_i = 0$. For us, $\Xi - \xi\xi$ is the part of D^2 with off-diagonal elements only, and it is easy to verify that it suffices to take a from $\mathbb{C}$ and b from $M_3(\mathbb{C})$ to satisfy the requirement $a\xi b = 0$, however then, one can generate elements of the bimodule with following (arbitrary) entries:

$$\begin{pmatrix} 0 & 0 & \bullet \\ 0 & 0 & 0 \\ \bullet & 0 & 0 \end{pmatrix}.$$

Therefore we can now ignore the off-diagonal entries, but for the diagonal part we have already established what kind of bimodule is generated by them (all considerations were independent of the $ab = 0$ condition).

A.1 Differential algebra for the tensor product with the manifold

A surprising feature of spectral triple noncommutative geometry is that tensoring two spectral triples can change the differential structure on the components. Here, the representation image of $\Omega^1(\mathcal{A}) \otimes_\mathcal{A} \Omega^1(\mathcal{A})$ on the Hilbert space intersects the image of the $\Omega^1(M) \otimes_\mathcal{A} \Omega^1(M)$ for any 4-dimensional manifold M. Therefore the differential ideal one has to quotient in order to obtain Ω^2 for the tensor product gets enlarged. Its restriction to the discrete component becomes just the image of the algebra $\mathcal{A}$ itself.

A.2 Scalar fields, gauge theory and the Connes-Lott action

We calculate here the discrete part of the gauge curvature and the total action for the Connes-Lott model. For the calculation of dA and AA we might restrict ourselves to the $ab = 0$ case, as in the case $ab \neq 0$ the possible additional terms would be off-diagonal and - as we already know, they would be in the subbimodule which one divides out.

The gauge potential is a self-adjoint one-form A, which we shall parametrize in the following way.

$$A = \begin{pmatrix} 0 & a(\Phi - 1) & 0 \\ (\Phi^* - 1)a^\dagger & 0 & \Psi - b \\ 0 & \Psi^\dagger - b^\dagger & 0 \end{pmatrix},$$

where Ψ is a 2×3 matrix and Φ is a quaternion. The shift in the parametrization is to simplify the formulas and use physical fields Φ and Ψ which transform homogeneously under gauge transformations.

Then using [8] we find:

$$dA = \begin{pmatrix} a(\Phi - 1)a^\dagger & 0 & a(\Psi - b) + a(\Phi - 1)b \\ 0 & a^\dagger a(\Phi - 1) + b(\Psi - b)^\dagger & 0 \\ 0 & 0 & b^\dagger(\Psi - b) \end{pmatrix} + \text{h.c.}$$

and for AA:

$$AA = \begin{pmatrix} a(\Phi - 1)^\dagger(\Phi - 1)a^\dagger & 0 & a(\Phi - 1)(\Psi - b) \\ 0 & (\Phi - 1)^* a^\dagger a(\Phi - 1) + (\Psi - b)(\Psi - b)^\dagger & 0 \\ (\Psi - b)^\dagger(\Phi - 1)^* a^\dagger & 0 & (\Psi - b)^\dagger(\Psi - b) \end{pmatrix},$$

Taking into account the form of Ω_2 for various situations we may now write the resulting Yang-Mills action $S = (F, F)$ for various situations:

- $a^\dagger a + bb^\dagger$ proportional to 1

    - $ab = 0$

$$S = a(|\Phi|^2 - 1)a^\dagger + a(\Psi\Phi)(\Psi\Phi)^\dagger a^\dagger + \text{Tr}\left|\Phi^* a^\dagger a\Phi - a^\dagger a + \Psi\Psi^\dagger - bb^\dagger\right|^2.$$

    - $ab \neq 0$

$$S = a(|\Phi|^2 - 1)a^\dagger + \text{Tr}\left|\Phi^* a^\dagger a\Phi - a^\dagger a + \Psi\Psi^\dagger - bb^\dagger\right|^2.$$

- $a^\dagger a + bb^\dagger$ not proportional to 1

    - $ab = 0$

$$S = a(|\Phi|^2 - 1)a^\dagger + a(\Psi\Phi)(\Psi\Phi)^\dagger a^\dagger + \left|\frac{1}{2}\text{Tr}\left(\Phi^* a^\dagger a\Phi - a^\dagger a + \Psi\Psi^\dagger - bb^\dagger\right)\right|^2.$$

    - $ab \neq 0$

$$S = a(|\Phi|^2 - 1)a^\dagger + \left|\frac{1}{2}\text{Tr}\left(\Phi^* a^\dagger a\Phi - a^\dagger a + \Psi\Psi^\dagger - bb^\dagger\right)\right|^2.$$

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Let us briefly describe the minima for the various situations. First, in all cases the action is a sum of positive terms. Thus we have to find the solutions of $S = 0$.

If $ab \neq 0$ the Higgs potential term $a(|\Phi|^2 - 1)a^\dagger$ requires Φ being unitary, by a gauge we can take $\Phi = 1$. The remaining term reduces to $\Psi\Psi^\dagger = bb^\dagger$ i.e $\Psi = bU^\dagger$, where $U \in U(3)$. Clearly this solution leads to spontaneous breaking of the color symmetry.

For $ab = 0$ the solution is the same, since then (if $\Phi = 1$, $\Psi = b$) the term $a\,(\Psi\Phi)$ vanishes. This is a consequence of a theorem that has been proven in [8].

A.3 Physical fields and spontaneous symmetry breaking

The action that we have calculated in the previous subsection is still preliminary, since only the discrete differential structure has been used. Taking into account the complete differential algebra, one has to take care about the enlarged differential ideal.

Recall that now we also have to quotient out the subbimodule,which is isomorphic to the algebra itself. If there were only one family of fermions, this would lead to a vanishing potential for the scalar fields. Adding families, we can assume that $b^\dagger b$ is of the form $B \otimes \mathrm{id}_{N_f}$. In other words, we assume that the leptoquarks couple diagonally to the fermion families. Phenomenologically this requirement is very attractive [6]. In this situation the $\Psi\Psi^\dagger$, $bb^\dagger$ are in the algebra and the potential reduces to

$$V(\Psi, \Phi) = V(\Phi) + a(\Psi\Phi)(\Psi\Phi)^\dagger a^\dagger,$$

(if $ab = 0$) and there is no symmetry breaking self-interaction term for Ψ.

Unfortunately, there is also no mass term for Ψ, except the one that comes from the expectation value of Φ. Thus, at least three components of the field Ψ will remain massless [5].

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[5]If neutrinos are massive, a will be a 2×2 matrix with eigenvalues m_e, m_ν. In this case all the components of Ψ would, at first sight, obtain masses from the coupling with the Higgs. However, since the rank of a will then be 2, there will be no nontrivial solution of $ab = 0$. Thus the leptoquarks and the Higgs will decouple completely.

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B The spectral action

The new *spectral principle* relates the bosonic part of the action with the eigenvalues of the square of the Dirac operator. It is defined by:

$$S_B = \text{Tr}\chi\left(\frac{\mathcal{D}^2}{\Lambda^2}\right),$$

and can easily be computed using the heat kernel technique [13]:

$$S_B = \frac{1}{16\pi^2}\int_M \left(\Lambda^4 f_0 a_0 + \Lambda^2 f_2 a_2 + f_4 a_4 + \Lambda^{-2} f_6 a_6 + \dots\right)\text{d}v(x). \quad (1)$$

Here f_i are the usual moments of the function χ, and the first three nonvanishing heat kernel coefficients $a_n(x, P)$ can directly be written down [13] if one casts P in the form:

$$P := \frac{\mathcal{D}^2}{\Lambda^2} = -\left(g^{\mu\nu}\partial_\mu\partial_\nu + A^\mu\partial_\mu + B\right) \quad (2)$$

We shall only retain the contributions from a_0, a_2 and a_4. Since the calculation is straightforward, we shall not give the complete expression for P and we shall only state the resulting bosonic action.

Before we do so, it is necessary to comment on the free parameters of the theory. They are Λ, f_0, f_2 and f_4. In the situation with $b = 0$, the operator $\mathcal{D}^2$ has a block diagonal form, acting on each of the three lepton families and the quarks separately. This offers the possibility to introduce four further parameters x, y_i by modifying the definition of the bosonic action as

$$S_B = x\text{Tr}_Q\chi\left(\frac{\mathcal{D}^2}{\Lambda^2}\right) + \sum_{i=1}^3 y_i\text{Tr}_i\chi\left(\frac{\mathcal{D}^2}{\Lambda^2}\right),$$

where Tr_i denotes the trace in the subspace spanned by the i-th lepton family, and Tr_Q is the trace over the subspace spanned by the quarks [6].

In our case with $b \neq 0$, there is no such decomposition of $\mathcal{H}$ since $\mathcal{D}^2$ mixes leptons and antiquarks. The only free parameters are therefore Λ, f_i. Additionally, we have still not identified the parameters of b with physical quantities. Let us assume that b is of the form

$$b = \tilde{b} \otimes \text{diag}(\kappa_e, \kappa_\mu, \kappa_\tau).$$

---

[6] Note that it is also possible to take four different functions χ_i, χ_Q

Then the leptoquark couples diagonally to the families, with coupling constants

$$k_i \sim \frac{\kappa_i}{\sqrt{\sum_i f_4 \kappa_i^2}}.$$

We shall write the components of the leptoquark as vectors $\vec{\psi}$. The resulting potential for the scalar fields (in the limit of flat spacetime) is then given as:

$$
\begin{aligned}
V(\vec{\psi}_1, \vec{\psi}_2, \phi) \;=\;& \alpha \left(|\vec{\psi}_1|^4 + |\vec{\psi}_2|^4 + 2|\vec{\psi}_1 \cdot \vec{\psi}_2|^2 \right) - \frac{\mu^2}{2} \left(|\vec{\psi}_1|^2 + |\vec{\psi}_2|^2 \right) \\
& + \delta |\vec{\psi}_1|^2 \phi^2 + \lambda \phi^4 - \frac{\mu^2}{2} \phi^2.
\end{aligned}
$$

The parameters of this potential are explicitly given as

$$
\begin{aligned}
\alpha &= \frac{2\pi^2 K_2}{f_4 K^2}, \\
\mu &= 2 \frac{f_2}{f_4} \Lambda^2, \\
\delta &= \frac{2\pi^2 M}{f_4 K L}, \\
\lambda &= \frac{\pi^2 L_2}{f_4 L^2},
\end{aligned}
$$

where

$$
\begin{aligned}
K_2 &= \sum_i \kappa_i^4, \\
K &= \sum_i \kappa_i^2, \\
L &= \sum_i (3 m_{u_i}^2 + 3 m_{d_i}^2 + m_{e_i}^2), \\
L_2 &= \sum_i (3 m_{u_i}^4 + 3 m_{d_i}^4 + m_{e_i}^4), \\
M &= \sum_i \kappa_i^2 (m_{u_i}^2 + m_{d_i}^2 + m_{e_i}^2),
\end{aligned}
$$

This potential leads to a nonvanishing vacuum expectation value of $|\vec{\psi}_2|^2$ and thus to spontaneous breaking of color symmetry.

References

[1] A.Connes, *Gravity coupled with matter and the foundation of noncommutative geometry*, Comm.Math.Phys, 155,109 (1996) A.Connes, *Noncommutative Geometry and reality*, J.Math.Phys. 36 (11) 1995

[2] L.Carminati, B.Iochum, T.Schucker, *Noncommutative Yang-Mills and noncommutative relativity: A Bridge over troubled water*, e-Print Archive: hep-th/9706105 D.Kastler, T.Schucker, *A Detailed account of Alain Connes' version of the standard model*, Rev.Math.Phys.8:205-228,1996

[3] B.Iochum, D.Kastler, T.Schucker, *Fuzzy mass relations in the standard model*, e-Print Archive: hep-th/9507150 B.Iochum, D.Kastler, T.Schucker, *Fuzzy mass relations for the Higgs*, J.Math.Phys.36:6232-6254, (1995)

[4] R.Coquereaux, *On the finite dimensional quantum group $M_3 + (M_{2|1}(\Lambda^2))_0$*, hep-th/9610114

[5] H1 Collaboration (C. Adloff et al.), *Observation of events at very high Q^2 in e p collisions at HERA*, Z.Phys.C74:191-206, (1997)

[6] K.S. Babu, Ch.Kolda, J.March-Russell, F.Wilczek, *Comments on the high Q^2 HERA anomaly*, Phys.Lett.B402:367-373, (1997) M.Leurer *A comprehensive study of leptoquark bounds*, Phys.Rev.D49,333,(1994)

[7] F. Lizzi, G. Mangano, G. Miele, G. Sparano, *Mirror fermions in noncommutative geometry*, e-Print Archive: hep-th/9704184

[8] M.Paschke, A.Sitarz, *Discrete spectral triples and their symmetries*, e-Print Archive: q-alg/9612029

[9] T.Krajewski, *Classification of finite spectral triples*, preprint CPT-96/P.3409

[10] A.Connes, Lott, *The Metric aspect of noncommutative geometry*, In:Cargese 1991, Proceedings, New symmetry principles in quantum field theory, 53-93.

[11] A.Connes, A.Chamseddine,*The Spectral Action Principle*, hep-th/9606001

[12] M.Paschke, A.Sitarz, *in preparation*

[13] P.B.Gilkey, *Invariance theory, the heat equation and the Atiyah-Singer index theorem*, CRC Press, (1995),